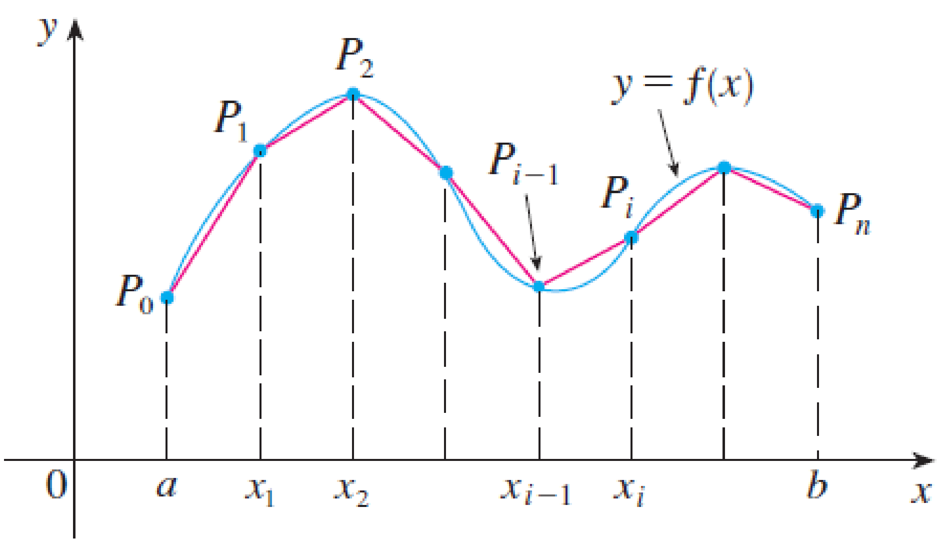
Closing Wed: HW_8A, 8B (8.1)
Midterm 2 will be returned Tuesday
No Class Wednesday – Use this time to work on homework or check out the final exam archive.

8.1 Arc Length

Goal: Given y = f(x) from x = a to x = b.
Want to find the ***length*** along the curve.

$$\text{Arc Length} = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$$



Derivation:

1. Break into *n* subdivision:
   $\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\Delta x$
2. Compute $y_i = f(x_i)$.

3. Compute the straight line distance from (x_i, y_i) to (x_{i+1}, y_{i+1}).

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

$$= \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y_i)^2}{(\Delta x)^2}\right)}$$

$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\ \Delta x$$

4. Add these distances up.

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\ \Delta x$$

Note that

$$\lim_{\Delta x \to 0} \frac{\Delta y_i}{\Delta x} = \text{slope of tangent} = f'(x)$$

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(f'(x)\right)^2}\ \Delta x$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Good news:
Arc length is important. And we found an integral to compute arc length, yeah!

Bad news:
The arc length integral almost always is something that can't be done explicitly (we have to approximate, *Simpson's rule*), boo!

In homework, you see the few, *unusual* cases where you actually can compute arc length explicitly.

Here are most of the 8.1 HW questions
Find the arc length of

1. $y = 4x - 5$ for $-3 \leq x \leq 2$.

2. $y = \sqrt{2 - x^2}$ for $0 \leq x \leq 1$.

3. $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ for $1 \leq x \leq 2$.

4. $y = \dfrac{1}{3}\sqrt{x}(x - 3)$ for $4 \leq x \leq 16$.

5. $y = \ln(\cos(x))$ for $0 \leq x \leq \pi/3$.

6. $y = \ln(1 - \mathrm{x}^2)$ for $0 \leq x \leq 1/7$.

Example:

$$y = 4x - 5 \text{ for } -3 \leq x \leq 2.$$

Example:

$$y = \frac{x^4}{8} + \frac{1}{4x^2} \quad \text{for } 1 \le x \le 2$$

Example:

$$y = \ln(\cos(x)) \text{ for } 0 \leq x \leq \pi/3.$$

Aside (don't need all this for this course)

In applications, Arc Length is used in motion (parametric) problems, which you will see a lot in Math 126:

$$x = x(t), y = y(t)$$

In this case, the same derivation from the beginning of class yields:

$$\text{Arc Length} = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

This gives the ***distance*** the object has traveled on the curve.

Very often, in motion problems we need:

$$s(t) = \int_0^t \sqrt{\left(x'(u)\right)^2 + \left(y'(u)\right)^2}\, du$$

which gives the distance traveled from time 0 to time t. This is called the ***Arc Length (Distance) Function***.

Simple Example:

Consider

$$x = 3t, y = 4t + 2$$

where t is in seconds.

(a) Find the arc length from 0 to 10 sec.

(b) Find the arc length function.

(c) What is the derivative of the arc length function?